Int J Theor Phys (2010) 49: 1422–1430
DOI 10.1007/s10773-010-0321-6

Coherent-State Path Integrals and Brownian Motions

Hideyasu Yamashita

Received: 8 January 2010 / Accepted: 15 March 2010 / Published online: 31 March 2010

Abstract A stochastic approach to the rigorous foundation of the coherent-state (phase-space) path integral is given. Stochastic integrals and some generalizations of the Feynman–Kac theorem are used for this purpose. In this approach, quantum mechanics is described in terms of the Fock–Bargmann representation; a classical Hamiltonian is related to the corresponding quantum Hamiltonian on the Fock–Bargmann space, seen as a Hilbert subspace of $L^2(\mathbf{R}^2)$. The coherent-state path integral is realized as a conditional expectation of a stochastic process defined by the exponential of the Fisk–Stratonovich integral of the fundamental 1-form along a path of Brownian motion on the phase space $\mathbf{R}^2$.

Keywords Path integral · Coherent state · Stochastic process · Brownian motion

1 Introduction

The essence of the method of the *coherent-state path integral* can be sketched by a simple example, the derivation of the coherent-state path integral representation for a harmonic oscillator (see e.g. Swanson [1]); For $\lambda \in \mathbf{C}$, let $|\lambda\rangle$ denote a coherent state. Let a^* and a be the creation and annihilation operators respectively, and the Hamiltonian operator H be

$$H = \omega\left(a^* a + \frac{1}{2}\right),$$

where $\omega \geq 0$. Let $|\lambda, t\rangle := e^{iHt}|\lambda\rangle$, and

$$\mathrm{d}\mu(\lambda) := \prod_{j=1}^{N-1} \frac{1}{2\pi i} \mathrm{d}\lambda_j^* \mathrm{d}\lambda_j.$$

H. Yamashita (✉)
Aichi-Gakuin University, Iwasaki-cho Araike 12, Nissin City, Aichi Prefecture, Japan
e-mail: yamasita@dpc.aichi-gakuin.ac.jp

Then by the overcompleteness we have

$$\langle\lambda_b, t_b|\lambda_a, t_a\rangle = \int \prod_{j=0}^{N-1} \langle\lambda_{j+1}, t_{j+1}|\lambda_j, t_j\rangle \mathrm{d}\mu(\lambda)$$

where $t_j = t_a + j(t_b - t_a)/N$, $j = 0, \dots, N$.

As a formal "continuum limit" $N \to \infty$ of the right-hand side, the path integral representation of the time evolution is given by

$$\langle\lambda_b, t_b|\lambda_a, t_a\rangle = e^{-(1/2)(\lambda_b^*\lambda_b - \lambda_a^*\lambda_a)} \int_{\lambda(t_a)=\lambda_a}^{\lambda(t_b)=\lambda_b} \mathcal{D}\lambda \exp\left\{ i \int_{t_a}^{t_b} \mathrm{d}t[i\lambda^*\dot{\lambda} - \omega\lambda^*\lambda] \right\}. \tag{1}$$

However, we find that it is difficult to give a mathematically rigorous meaning to the right-hand side above, even in the physically trivial case where $\omega = 0$. But note that for an absolutely continuous path λ, the following equation holds:

$$\int_{t_a}^{t_b} \lambda^*\dot{\lambda}\mathrm{d}t = \int_{t_a}^{t_b} \lambda^*\mathrm{d}\lambda,$$

where $\dot{\lambda}$ is the derivative of λ in the sense of Radon–Nikodym, and the right-hand side is the integral in the sense of Lebesgue–Stieltjes, which is defined when λ has a bounded variation. A Stieltjes-type integral can be defined for a sort of path which does not has a bounded variation, such as a Brownian motion. Such integral is called the *stochastic integral*. In this paper we suggest a stochastic approach to this problem; Here λ is interpreted as a Brownian motion, the integral measure $\mathcal{D}\lambda$ as the probability measure on the path space, and the above integral as a stochastic integral.

In this paper we use the Feynman–Kac theorem and its generalizations for this purpose. However our approach differs from the conventional Feynman–Kac approaches to path integrals in the following point: Usually the Feynman–Kac theorem is used for rigorous treatments of *Euclidean* (*i.e. imaginary-time*) *Lagrangian path integrals*, where the paths are on the classical configuration space. On the other hand, we use it for that of *real-time Hamiltonian path integrals,* where the paths are on the classical phase space.

2 Fock–Bargmann Space

First we review the definitions and properties on the *Fock–Bargmann representation* (see e.g. [2]).

We identify $\mathbf{R}^2$ with $\mathbf{C}$ by $z = x + iy$ $(x, y \in \mathbf{R})$. Define the differential operators $\partial_x, \partial_y, \partial_z$ and $\partial_{\bar{z}}$ on $L^2(\mathbf{C}) = L^2(\mathbf{R}^2)$ by

$$\partial_x := \frac{\partial}{\partial x}, \qquad \partial_y := \frac{\partial}{\partial y}, \qquad \partial_z := \frac{1}{2}(\partial_x - i\partial_y), \qquad \partial_{\bar{z}} := \frac{1}{2}(\partial_x + i\partial_y),$$

and the linear operators $\mathbf{z}, \bar{\mathbf{z}}$ on $L^2(\mathbf{C})$ by

$$(\mathbf{z}f)(z) := zf(z), \qquad (\bar{\mathbf{z}}f)(z) := \bar{z}f(z).$$

Define the subspace $\mathcal{H}_{\mathrm{FB}} \subset L^2(\mathbf{R}^2)$ by

$$\mathcal{H}_{\mathrm{FB}} := L^2(\mathbf{R}^2) \cap \{e^{-(1/2)\bar{\mathbf{z}}\mathbf{z}} f | f \in C^\infty(\mathbf{R}^2),\ \partial_{\bar{z}} f = 0\}. \tag{2}$$

The condition $\partial_{\bar{z}} f = 0$ means that $f : \mathbf{C} \to \mathbf{C}$ is entire analytic. The subspace $\mathcal{H}_{\mathrm{FB}}$ is shown to be a Hilbert space. Let us call $\mathcal{H}_{\mathrm{FB}}$ the *Fock–Bargmann space* (*FB space*). We easily see that $\mathcal{H}_{\mathrm{FB}}$ have another expression:

$$\mathcal{H}_{\mathrm{FB}} = \{f \in L^2(\mathbf{R}^2) \cap C^\infty(\mathbf{R}^2) | (\partial_{\bar{z}} + \mathbf{z}/2) f = 0\} = \mathrm{Ker}(\partial_{\bar{z}} + \mathbf{z}/2).$$

A orthonormal basis of $\mathcal{H}_{\mathrm{FB}}$ is given by

$$e_n(z) := e^{-(1/2)\bar{z}z} \frac{z^n}{\sqrt{n!}}, \quad n = 0, 1, \dots,$$

in other words,

$$e_0(z) := e^{-(1/2)\bar{z}z}, \qquad e_n = \frac{\mathbf{z}}{\sqrt{n}} e_{n-1},$$

which is equivalent to the number-state representation of a Bose harmonic oscillator, where $\mathbf{z}$ is seen as the *creation operator*.

Let $\mathbf{z}^*$ denote the adjoint operator of $\mathbf{z}$ on $\mathcal{H}_{\mathrm{FB}}$. Note that this is different from $\bar{\mathbf{z}}$, the adjoint operator of $\mathbf{z}$ on $L^2(\mathbf{R}^2)$:

$$\mathbf{z}^* f = E_{\mathcal{H}_{\mathrm{FB}}} \bar{\mathbf{z}} f = (\bar{\mathbf{z}}/2 + \partial_z) f, \quad f \in \mathcal{H}_{\mathrm{FB}}$$

where $E_{\mathcal{H}_{\mathrm{FB}}}$ is the orthogonal projection from $L^2(\mathbf{R}^2)$ onto $\mathcal{H}_{\mathrm{FB}}$. The operator $\mathbf{z}^*$ is called the *annihilation operator* on $\mathcal{H}_{\mathrm{FB}}$.

For $\alpha \in \mathbf{C}$, define the operator $D(\alpha)$ on $\mathcal{H}_{\mathrm{FB}}$ by

$$D(\alpha) := \exp[\alpha \mathbf{z} - \bar{\alpha} \mathbf{z}^*].$$

This is a unitary representation of the Heisenberg group, called the *Fock–Bargmann representation.* The action of $D(\alpha)$ is explicitly expressed by

$$D(\alpha) f(z) = e^{(1/2)[\alpha z - \bar{\alpha}\bar{z}]} f(z - \bar{\alpha}) = e^{\mathrm{i}\Im(\alpha z)} f(z - \bar{\alpha}), \quad f \in \mathcal{H}_{\mathrm{FB}}$$

For $\alpha \in \mathbf{C}$, define $C_\alpha \in \mathcal{H}_{\mathrm{FB}}$, called a *coherent state,* by

$$C_\alpha(z) := \exp[-(1/2)(\bar{z}z + \bar{\alpha}\alpha + 2\alpha z)]. \tag{3}$$

We see

$$C_\alpha = D(\alpha) C_0 = D(\alpha) e_0,$$

and

$$\langle C_\xi | C_\zeta \rangle = \exp\left[-\frac{1}{2}(\bar{\xi}\xi - 2\bar{\xi}\zeta + \bar{\zeta}\zeta)\right] = e^{-(1/2)\bar{\zeta}\zeta} C_{\bar{\xi}}(\zeta).$$

Let $E_\alpha = |C_\alpha\rangle\langle C_\alpha|$, then we have

$$\int E_\alpha \mathrm{d}\mu(\alpha) = E_{\mathcal{H}_{\mathrm{FB}}}, \qquad \mathrm{d}\mu(z) := \pi^{-1} \mathrm{d}x \mathrm{d}y,$$

where $E_{\mathcal{H}_{\mathrm{FB}}}$ is the orthogonal projection from $L^2(\mathbf{R}^2)$ onto $\mathcal{H}_{\mathrm{FB}}$. This property is called the *overcompleteness* of the coherent states on $\mathcal{H}_{\mathrm{FB}}$. Clearly the action of $E_{\mathcal{H}_{\mathrm{FB}}}$ on $f \in L^2(\mathbf{R}^2)$

is expressed as follows:

$$E_{\mathcal{H}_{\mathrm{FB}}} f(z) = \int \exp\left[-\frac{1}{2}(\bar{z}'z' - 2\bar{z}'z + \bar{z}z)\right] f(z')\mathrm{d}\mu(z').$$

Define the self-adjoint operators D_x, D_y and $\mathcal{A}$ on $L^2(\mathbf{R}^2) \cong L^2(\mathbf{C})$ by

$$D_x := i\partial_x - y, \qquad D_y = i\partial_y + x, \tag{4}$$

$$\mathcal{A} := D_x^2 + D_y^2 = -\Delta + 2i(-y\partial_x + x\partial_y) + x^2 + y^2 \tag{5}$$

We can check the relation $[D_x, D_y] = 2i$, the (reducible) canonical commutation relation with $\hbar = 2$. Define the corresponding "annihilation" and "creation" operators $a_{\mathrm{FB}}, a_{\mathrm{FB}}{}^*$ respectively by

$$a_{\mathrm{FB}} := \frac{1}{2}(D_y - iD_x), \qquad a_{\mathrm{FB}}{}^* := \frac{1}{2}(D_y + iD_x).$$

Define the "number operator" by $N_{\mathrm{FB}} = a_{\mathrm{FB}}{}^* a_{\mathrm{FB}}$, whose spectrum is $\{0, 1, 2, \ldots\}$. Then we find

$$\mathcal{A} = 4N_{\mathrm{FB}} + 2.$$

Note that

$$a_{\mathrm{FB}} = \frac{1}{2}(\partial_x + i\partial_y + x + iy) = \partial_{\bar{z}} + \mathbf{z}/2$$

where $z = x + iy$. Thus we find

$$\mathrm{Ker}\, N_{\mathrm{FB}} = \mathrm{Ker}\, a_{\mathrm{FB}} = \{f \in L^2(\mathbf{R}^2) \cap C^\infty(\mathbf{R}^2) | (\partial_{\bar{z}} + \mathbf{z}/2)f = 0\},$$

which equals the Fock–Bargmann space $\mathcal{H}_{\mathrm{FB}} \subset L^2(\mathbf{R}^2)$. Thus we have another representation of $E_{\mathcal{H}_{\mathrm{FB}}}$:

$$E_{\mathcal{H}_{\mathrm{FB}}} = \lim_{t\to\infty} \exp(-tN_{\mathrm{FB}}) \quad \text{(in norm)}.$$

2.1 Quantization on a FB Space

Let A and B be possibly unbounded positive self-adjoint operators on a Hilbert space $\mathcal{H}$. Suppose the following spectral conditions:

$$0 \in \mathrm{spec}(A), \quad \mathrm{spec}(A) \cap (0, \alpha) = \emptyset \quad \text{for some } \alpha > 0.$$

Fix $\sigma > 0$, and suppose that $\{f_{\sigma,t} \in \mathrm{dom}(A) \cap \mathrm{dom}(B) | \sigma > 0, t \geq 0\}$ satisfy the following differential equation

$$(\mathrm{d}/\mathrm{d}t) f_{\sigma,t} = (-\sigma^2 A + iB) f_{\sigma,t}, \quad f_{\sigma,0} = f.$$

Let E be the orthogonal projection onto $\mathrm{Ker}\, A$.

Lemma 1 *If B is bounded and $(I - E)f = 0$, then for all $t \geq 0$,*

$$\|(I - E)f_{\sigma,t}\| \leq \frac{1}{\alpha\sigma^2}\|B\|.$$

Proof We have

$$
\begin{aligned}
(\mathrm{d}/\mathrm{d}t)\|(I-E)f_{\sigma,t}\|^2 &= (\mathrm{d}/\mathrm{d}t)\langle f_{\sigma,t}|(I-E)f_{\sigma,t}\rangle \\
&= \langle(-\sigma^2A+iB)f_{\sigma,t}|(I-E)f_{\sigma,t}\rangle + \langle f_{\sigma,t}|(I-E)(-\sigma^2A+iB)f_{\sigma,t}\rangle \\
&= \langle f_{\sigma,t}|[(-\sigma^2A-iB)(I-E)+(I-E)(-\sigma^2A+iB)]f_{\sigma,t}\rangle \\
&= \langle f_{\sigma,t}|[-2\sigma^2A+i[(I-E),B]]f_{\sigma,t}\rangle \\
&= -2\sigma^2\langle f_{\sigma,t}|Af_{\sigma,t}\rangle + i\langle f_{\sigma,t}|[(I-E),B]f_{\sigma,t}\rangle \\
&= -2\sigma^2\langle f_{\sigma,t}|Af_{\sigma,t}\rangle - 2\Im\langle f_{\sigma,t}|(I-E)Bf_{\sigma,t}\rangle.
\end{aligned}
$$

Since $0 \le \alpha(I-E) \le A$, we have

$$
\begin{aligned}
(\mathrm{d}/\mathrm{d}t)\|(I-E)f_{\sigma,t}\|^2 &\le -2\alpha\sigma^2\langle f_{\sigma,t}|(I-E)f_{\sigma,t}\rangle - 2\Im\langle f_{\sigma,t}|(I-E)Bf_{\sigma,t}\rangle \\
&\le -2\alpha\sigma^2\|(I-E)f_{\sigma,t}\|^2 + 2\|f\|\|B\|\|(I-E)f_{\sigma,t}\| \\
&= 2\left(-\alpha\sigma^2\|(I-E)f_{\sigma,t}\| + \|f\|\|B\|\right)\|(I-E)f_{\sigma,t}\|
\end{aligned}
$$

Let $h(t) := \|(I-E)f_{\sigma,t}\|^2$. Then we have shown that for all $t > 0$,

$$(\mathrm{d}/\mathrm{d}t)h(t) \ge 0 \quad \Rightarrow \quad h(t) \le \frac{\|f\|^2\|B\|^2}{\alpha^2\sigma^4}.$$

Since $h(0) = 0$, this implies $h(t) \le \frac{\|f\|^2\|B\|^2}{\alpha^2\sigma^4}$ for all $t \ge 0$. □

Suppose $(I-E)f_0 = 0$. Let $g_t := Ef_{\sigma,t}$, then we have $g_0 = f_0$ and

$$\|(\mathrm{d}/\mathrm{d}t)g_t - iEBEg_t\| = \|iEB(I-E)f_{\sigma,t}\| \le \|B\|\|(I-E)f_{\sigma,t}\| \le \frac{\|B\|^2}{\alpha\sigma^2}$$

for all $t > 0$.

$$\lim_{\sigma\to\infty} g_t = \exp(itEBE)g_0$$

Thus by Lemma 1, we have the following:

Theorem 1 *If B is bounded and $(I-E)f = 0$, then*

$$\forall t > 0, \quad \lim_{\sigma\to\infty} f_{\sigma,t} = \exp(itEBE)f.$$

Let $H_{\mathrm{C}} \in L^\infty(\mathbf{R}^2)$ be real, and set $\mathcal{H} = \mathcal{H}_{\mathrm{FB}}$, $A = 2N_{\mathrm{FB}} = (\mathcal{A}-2)/2$ and $B = H_{\mathrm{C}}$ (as a multiplication operator on $L^2(\mathbf{R}^2)$).

Corollary 1 *Let $f \in \mathcal{H}_{\mathrm{FB}}$ and suppose $\{f_{\sigma,t} \in L^2(\mathbf{R}^2)|\sigma > 0, t \ge 0\}$ satisfy the equation*

$$(\mathrm{d}/\mathrm{d}t)f_{\sigma,t} = (-2\sigma^2N_{\mathrm{FB}} + iH_{\mathrm{C}})f_{\sigma,t}, \quad f_{\sigma,0} = f. \tag{6}$$

Then the following holds for all $t > 0$:

$$\lim_{\sigma\to\infty} f_{\sigma,t} = \exp(itH)f, \tag{7}$$

where $H = E_{\mathcal{H}_{\mathrm{FB}}}H_{\mathrm{C}}E_{\mathcal{H}_{\mathrm{FB}}}$.

If we regard H_C as a *classical Hamiltonian* defined on the phase space $\mathbf{R}^2$, the operator H can be viewed as a *quantization* of H_C. Thus we can interpret the above corollary as a mathematical description of the quantization procedure, although here we are confining ourselves to the cases where both H_C and H are bounded.

3 Stochastic Representation

In this section we give a stochastic representation of the quantum time evolution on a FB space, along the lines of Corollary 1. We first review the *Feynman–Kac theorem.*

Let d be a positive integer, and consider a standard d-dimensional Brownian motion $B_t = (B_t^1, \ldots, B_t^d)$ $(t \geq 0)$. We suppose that the distribution of B_0 has the probability density function ρ such that $\rho(x) > 0$ for each $x \in \mathbf{R}^d$, so that the conditional expectation $E[X|B_0 = x]$ makes sense.

A function

$$v : [0, T] \times \mathbf{R}^d \to \mathbf{R}$$

is *of type* $C^{1,2}((0, T) \times \mathbf{R}^d)$ if the derivatives

$$\partial v/\partial t,\ \partial v/\partial x^i,\ \partial^2 v/\partial x^i \partial x^j, \quad 1 \leq i, j \leq d$$

are continuous on $(0, T) \times \mathbf{R}^d$.

Assume that $f : \mathbf{R}^d \to \mathbf{R}$, $V : \mathbf{R}^d \to [0, \infty)$ are continuous, and that the continuous function $u : [0, \infty) \times \mathbf{R}^d \to \mathbf{R}$ is of class $C^{1,2}$ on $(0, \infty) \times \mathbf{R}^d$ and satisfies

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u - Vu, \quad u(0, x) = f(x). \tag{8}$$

Theorem 2 (Feynman–Kac Theorem) *If $u(t, x)$ is polynomially increasing with respect to x, and if V is bounded, then u admits the stochastic representation*

$$u(t, x) = E\left[f(B_t) \exp\left\{ -\int_0^t V(B_s)\mathrm{d}s \right\} | B_0 = x \right] \tag{9}$$

Proof Let $v(t, x) := u(T - t, x)$ and define the processes I_t and M_t by

$$I_t := \int_0^t V(B_s)\mathrm{d}s, \qquad M_{t'} := v(t + t', B_{t'})e^{-I_{t'}}.$$

Then by Itô's rule and (8), we have $\mathrm{d}M_{t'} = \exp(-I_{t'})\nabla v(t + t', B_{t'}) \cdot \mathrm{d}B_{t'}$, and hence $v(t, x) = E[v(t + t', B_{t'}) \exp(-I_{t'}) | B_0 = x]$. Set $t' = T - t$ and we have (9). For further information, see [3]. □

Let $\int_0^t X \circ \mathrm{d}Y$ denote the *Fisk–Stratonovich integral*, defined by

$$\int_0^t X_s \circ \mathrm{d}Y_s := \int_0^t X_s \mathrm{d}Y_s + 2^{-1}\langle X^M, Y^M \rangle_t,$$

which is formally written as

$$X_t \circ \mathrm{d}Y_t := X_t \mathrm{d}Y_t + 2^{-1}\mathrm{d}X_t \mathrm{d}Y_t.$$

Let $\theta_k : \mathbf{R}^d \to \mathbf{R}$ ($k = 1, \dots, d$) be smooth and tempered (i.e. polynomially increasing), and $\theta = (\theta_1, \dots, \theta_d)$. Then the integral

$$I_t := \sum_{k=1}^{d} \int_0^t \theta_k(B_s) \circ \mathrm{d}B_s^k,$$

exists. We simply write $I_t = \int_0^t \theta(B_s) \circ \mathrm{d}B_s$. By Itô's rule, we see that this is expressed by the Itô stochastic integral as

$$I_t = \int_0^t \theta(B_s)\mathrm{d}B_s + \frac{\sigma^2}{2} \int_0^t (\mathrm{div}\theta)(B_s)\mathrm{d}s$$

where $\mathrm{div}\theta = \nabla \cdot \theta = \sum_k (\mathrm{d}/\mathrm{d}x^k)\theta_k$. If we regard θ as a 1-form on $\mathbf{R}^d$ (i.e. $\theta = \sum_k \theta_k \mathrm{d}x^k$), then I_t can be seen as a *stochastic line integral* [4] of the 1-form θ, and is written as

$$I_t = \int_{B[0,t]} \theta.$$

This is the natural extension of the usual notion of the (Lebesgue–Stieltjes) line integral $\int_C \theta$ in the case where the path $C : [0, t] \to \mathbf{R}^d$ has a bounded variation.

Theorem 3 *Define the operator* $\mathcal{D}(\theta)$ *by*

$$\mathcal{D}(\theta) := (i\nabla + \theta)^2 = \sum_{k=1}^{d} \left(i\frac{\partial}{\partial x^k} + \theta_k \right)^2 = -\Delta + 2i\theta \cdot \nabla + i(\nabla \cdot \theta) + \theta \cdot \theta. \tag{10}$$

Let $V : \mathbf{R}^d \to \mathbf{C}$ *be continuous and bounded. Suppose that* $u(t, x)$ *satisfy the differential equation*

$$\dot{u}(t, x) = -\frac{1}{2}\mathcal{D}(\theta)u(t, x) - V(x)u(t, x), \qquad u(0, x) = f(x), \tag{11}$$

and that $u(t, x)$ *is polynomially increasing w.r.t.* x. *Then* u *admits the stochastic representation*

$$u(t, x) = E\left[f(B_t) \exp(i\mathcal{S}_t^-) | B_0 = x \right] \tag{12}$$

where

$$\mathcal{S}_t^- = \mathcal{S}_t^-(B, \theta, V) := -\int_{B[0,t]} \theta + i \int_0^t V(B_s)\mathrm{d}s.$$

Proof By the argument similar to the proof of Theorem 2; Let $v(t, x) := u(T - t, x)$. Define the process $M_{t'}$ by

$$M_{t'} := v(t + t', B_{t'}) \exp[I_{t'} + J_{t'}]$$

where

$$I_{t'} := -\int_0^{t'} V(B_s)\mathrm{d}s, \qquad J_{t'} := -i \int_0^{t'} \theta(B_s) \circ \mathrm{d}B_s.$$

Then by Itô's rule and (11), we have $E^x[M_{t'}] = v(t, x)$. Set $t' = T - t$ and we have (12). □

Corollary 2 *Suppose the conditions of Theorem 3. Let $\sigma > 0$ and suppose that $u(t,x)$ satisfy the differential equation*

$$\dot{u}(t,x) = -\frac{\sigma^2}{2}\mathcal{D}(\theta)u(t,x) - V(x)u(t,x), \qquad u(0,x) = f(x), \tag{13}$$

and that $u(t,x)$ is polynomially increasing w.r.t. x. Then u admits the stochastic representation

$$u(t,x) = E[f(\sigma B_t)\exp(i\mathcal{S}_t^-(\sigma B,\theta,V))|\sigma B_0 = x]. \tag{14}$$

Proof Let $w(t,x) := u(\sigma^{-2}t, x)$, then we have

$$(\partial/\partial t)w(t,x) = (-(1/2)\mathcal{A}(\theta) - \sigma^{-2}V)w(t,x).$$

We can check $\mathcal{S}^-_{\sigma^2 t}(B,\theta,\sigma^{-2}V) = \mathcal{S}^-_t(C,\theta,V)$ with $C_t := B_{\sigma^2 t}$. Thus by Theorem 3, we have

$$\begin{aligned}u(t,x) = w(\sigma^2 t, x) &= E[f(B_{\sigma^2 t})\exp(i\mathcal{S}^-_{\sigma^2 t}(B,\theta,\sigma^{-2}V))|B_0 = x]\\ &= E[f(C_t)\exp(i\mathcal{S}^-_t(C,\theta,V))|C_0 = x]\\ &= E[f(\sigma B_t)\exp(i\mathcal{S}^-_t(\sigma B,\theta,V))|\sigma B_0 = x].\end{aligned}$$

□

For $t \ge 0$ and $y \in \mathbf{R}^d$, define $p(t,y)$ by

$$p(t,x) := \frac{1}{(2\pi\sigma^2 t)^{d/2}}\exp(-\|x\|^2/(2\sigma^2 t)).$$

For $0 \le t_1 < t_2$, the probability density function of $B_{t_2} - B_{t_1}$ equals $p(t_2 - t_1, \cdot)$. Note that

$$E[f(B_t)\exp(i\mathcal{S}_t^-)|B_0 = x] = \int_{\mathbf{R}^d} E[\exp(i\mathcal{S}_t^-)|B_0 = x, B_t = y]p(t, y-x)f(y)\mathrm{d}y,$$

and

$$E[\exp(i\mathcal{S}^+_{[t,t']})|B_t = x, B_{t'} = y] = E[\exp(i\mathcal{S}^-_{[t,t']})|B_{t'} = x, B_t = y]$$

where

$$\mathcal{S}^{\pm}_{[t,t']} := \pm\int_{B[t,t']}\theta + i\int_t^{t'} V(B_s)\mathrm{d}s. \tag{15}$$

for $0 \le t < t'$. Thus we have the following:

Corollary 3 *Suppose the conditions of Theorem 3 and (11). Let $f_t(x) = u(t,x)$, then the mapping $f_t \mapsto f_{t'}$ $(0 \le t < t')$ has the integral representation*

$$f_{t'}(x') = \int_{\mathbf{R}^d} K_\sigma(t,t';x,x')f_t(x)\mathrm{d}x, \tag{16}$$

where

$$K_\sigma(t,t';x,x') := E[\exp(i\mathcal{S}^+_{[t,t']}(\sigma B,\theta,V))|B_{t'} = x', B_t = x]p(t'-t, x'-x). \tag{17}$$

Suppose that there exists a solution $\{f_{\sigma,t} \in L^2(\mathbf{R}^2) | \sigma > 0, t \geq 0\}$ of (6). We see

$$-2\sigma^2 N_{\mathrm{FB}} + iH_{\mathrm{C}} = -\sigma^2\left(\frac{1}{2}\mathcal{A} - 1\right) + iH_{\mathrm{C}} = -\frac{\sigma^2}{2}\mathcal{D}(\theta) + iH_{\mathrm{C}} + \sigma^2$$

where $\mathcal{D}(\theta)$ is defined by (10), with $d = 2$, $\theta_1(x, y) = -y$ and $\theta_2(x, y) = x$. Thus, letting $V = -(iH_{\mathrm{C}} + \sigma^2)$ on (15), the following theorem follows from Corollaries 1 and 3:

Theorem 4 (Coherent-state path integral) *Let H be a self-adjoint operator on $\mathcal{H}_{\mathrm{FB}}$. If there exists a continuous bounded function $H_{\mathrm{C}} : \mathbf{R}^2 \to \mathbf{R}$ such that $H = E_{\mathcal{H}_{\mathrm{FB}}} H_{\mathrm{C}} \restriction \mathcal{H}_{\mathrm{FB}}$, then for $f \in \mathcal{H}_{\mathrm{FB}}$,*

$$\exp(itH)f(z) = \lim_{\sigma\to\infty} \pi \int_{\mathbf{C}} K_\sigma(z, z'; t) f(z') \mathrm{d}\mu(z') \quad (\textit{in } L^2(\mathbf{R}^2))$$

where

$$\begin{aligned}
K_\sigma(z, z'; t) &:= E[\exp(i\mathcal{S}^+_{\sigma,t}) | B_{\sigma,0} = z, B_{\sigma,t} = z'] p(t, z' - z), \\
B_{\sigma,t} &:= \sigma B_t \\
\mathcal{S}^+_{\sigma,t} &:= \int_{B[0,t]} \theta + \int_0^t H_{\mathrm{C}}(B_{\sigma,s})\mathrm{d}s - i\sigma^2 t, \\
\theta(x, y) &:= -y\mathrm{d}x + x\mathrm{d}y.
\end{aligned}$$

Note that the explicit expression of the above line integral is as follows:

$$\int_{B[0,t]} \theta = \int_0^t (-B^2_{\sigma,s} \circ \mathrm{d}B^1_{\sigma,s} + B^1_{\sigma,s} \circ \mathrm{d}B^2_{\sigma,s}).$$

This 1-form θ can be regarded as a *fundamental* 1-*form* (in the sense of classical analytical dynamics) on the phase space $\mathbf{R}^2$. Thus the process $\mathcal{S}^+_{\sigma,t}$ can be seen as a classical-mechanical quantity analogous to the action integral along the Brownian motion B.

References

1. Swanson, M.S.: Path Integrals and Quantum Processes. Academic Press, New York (1996)
2. Perelomov, A.: Generalized Coherent States and Their Applications. Springer, Berlin (1986)
3. Karatzas, I., Shreve, S.E.: Brownian Motion and Stochastic Calculus. Springer, Berlin (1991)
4. Hsu, E.P.: Stochastic Analysis on Manifolds. Am. Math. Soc., Providence (2002)